Introduction to Mathematical Morphology

Overview and trends

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Outline

- Historical notes
- Definitions and geometric interpretation
- Algebraic foundations
- Tools
- Current trends

Historical notes

Matheron and Serra: first work on Mathematical Morphology in 1976.

- Used in the context of stereology
- Two “schools”: american and french (prior to 1990)
- Work on fast algorithms (1990)

Major ideas

"Classical" signal processing theory / vector spaces:

- linearity is assumed
- pointwise operators: $f(x) \to g(x)$

Mathematical morphology:

- models are based on shapes and patterns
- essential objects are sets (not points), operators, graphs, trees

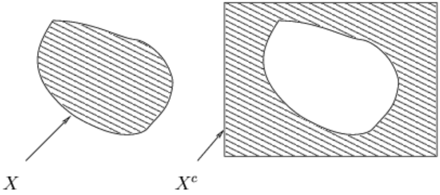
Framework: set theory

Set are denoted by $A, B, \dots$ and elements by $a, b, \dots$

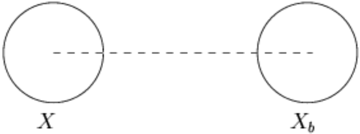
- **Equality**
  $X = Y \Leftrightarrow (x \in X \Rightarrow x \in Y \text{ and } x \in Y \Rightarrow x \in X)$.
- **Inclusion**
  $X \subseteq Y \Leftrightarrow (x \in X \Rightarrow x \in Y)$.
- **Intersection**
  $X \cap Y = \{x \text{ such that } x \in X \text{ and } x \in Y\}$.
- **Union**
  $X \cup Y = \{x \text{ such that } x \in X \text{ or } x \in Y\}$.

Additional operations

- **Complementary**
  $X^c = \{x \text{ such that } x \in \mathscr{E} \text{ and } x \notin X\}.$



- **Symmetric**
  $\check{X} = \{-x | x \in X\}.$
- **Translate**
  $X_b = \{z \in \mathscr{E} | z = x + b, x \in X\}.$



Basic operations on sets

Let $\mathscr{E}$ be a referentiel (for example $\mathbb{R}^n$ or $\mathbb{Z}^n$, with $n \geq 1$), a set $X \subseteq \mathscr{E}$ and a vector (or location) $b \in \mathscr{E}$,

Definition

Dilation

$$X \oplus B = \bigcup_{b \in B} X_b = \bigcup_{x \in X} B_x = \{x + b | x \in X, b \in B\}$$

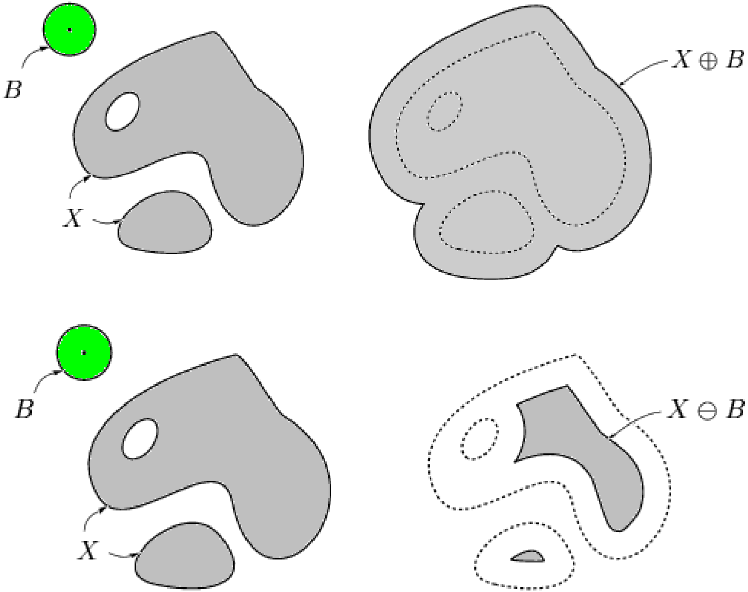
Definition

Erosion

$$X \ominus B = \bigcap_{b \in B} X_{-b} = \{p \in \mathscr{E} | B_p \subseteq X\}$$

Dilation and erosions are dual operators: $X \ominus \check{B} = (X^c \oplus B)^c$

Illustrations



Cascading operators

Definition

Opening

$$X \circ B = (X \ominus B) \oplus B \tag{1}$$

Geometric interpretation:

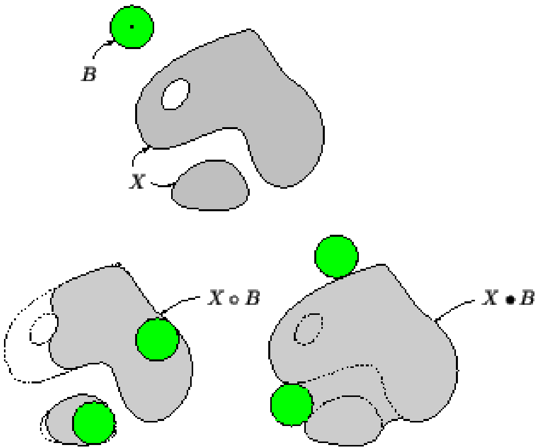
$$X = \bigcup \{B_p | B_p \subseteq X\} \tag{2}$$

Definition

Closing

$$X \bullet B = (X \oplus B) \ominus B \tag{3}$$

Illustrations



Properties on operators

Increasingness: ordering is preserved:

If $X \subseteq Y$, then $(X \ominus B) \subseteq (Y \ominus B)$ and $(X \oplus B) \subseteq (Y \oplus B)$;

Anti-extensivity / extensivity: shrinking or expanding

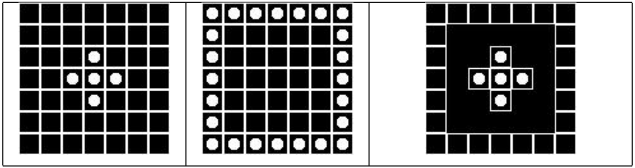
If the origin belongs to B ($o \in B$): $X \ominus B \subseteq X$ and $X \subseteq X \oplus B$

Idempotence: more or less the notion of ideal linear filter

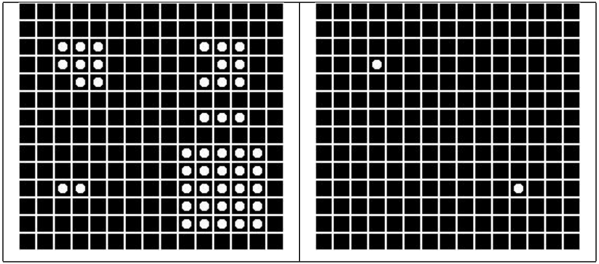
$(X \circ B) \circ B = (X \circ B)$ and $(X \bullet B) \bullet B = (X \bullet B)$

“Quest” for these properties

Hit or Miss transform



Hit	Miss	Hit or Miss pattern

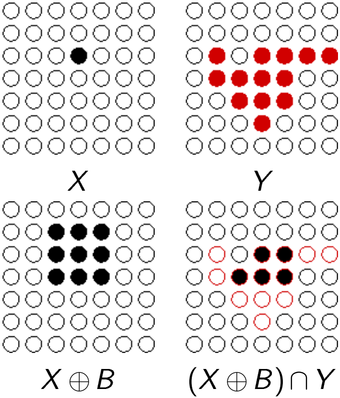


Original	Result

Reconstruction

Take $X \subseteq Y$ (Y is a mask). Repeat the following operation (geodesic dilation):

$$(X \oplus B) \cap Y$$



Notes on algebraic properties

- Most properties valid for sets are applicable to other data structures: grayscale images, color images, graphs, trees.
- The order is tricky:
  - how do we order RGB values?
  - order is not complete, to the contrary of numbers
    If N, M are numbers, then $N \leq M$ or $N > M$.
    For two functions f, g: $f(x) \leq g(x)$ or $f(x) > g(x)$ for $x \in D$, a subset of $\mathbb{R}^2$.
  - $\Rightarrow$Algebraic notions of *partial order*, *complete lattices*, ...
- There are dual concepts

Functions

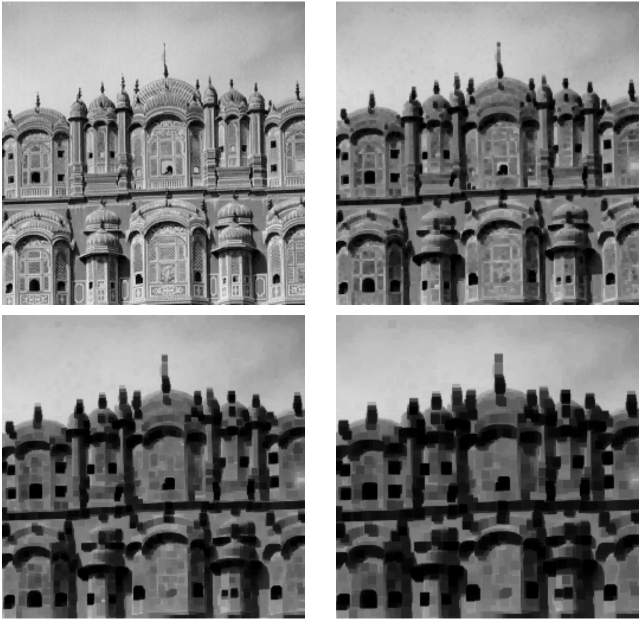
Switch from binary sets to functions:

- Replace the union $\cup$ by the supremum $\vee$
- Likewise, the intersection $\cap$ by the infimum $\wedge$
- Define the complementary set: $f^c(x) = 255 - f(x)$
- The (horizontal) translate: $f_b(x) = f(x - b)$

and there you go:

$$\varepsilon_B(f) = f \ominus B = \bigwedge_{b \in B} f_{-b} \qquad \delta_B(f) = f \oplus B = \bigvee_{b \in B} f_b \qquad \gamma_B = \delta_B \varepsilon_B \ldots$$

Erosion



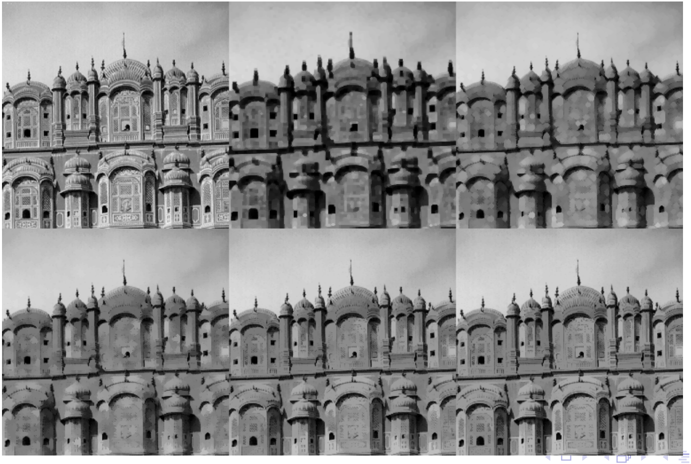
Opening

Grayscale reconstruction

Example

Original, eroded image and successive geodesic dilations:

$$(((\varepsilon_B(f) \oplus B) \wedge f) \oplus B) \wedge f \ldots$$



A note on algorithms

- Large structuring elements.
  Size of a structuring element: a *set B of size n*, denoted nB, is usually defined as

$$nB = \underbrace{B \oplus B \oplus \ldots \oplus B}_{n-1 \text{ dilations}} \tag{4}$$

- The definition of an operation usually leads to the worst implementation. Really!
- Useful property (chain rule):
  - $X \ominus (nH \oplus mV) = (X \ominus nH) \ominus mV$
- Logarithmic decomposition.
  For example, if $\partial(B)$ denotes the border of B,

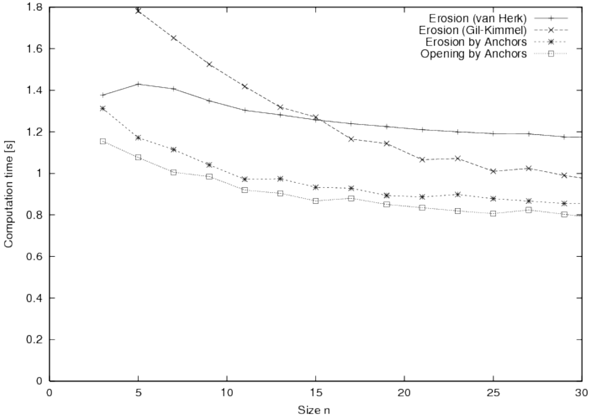
$$9B = B \oplus \partial(B) \oplus \partial(B) \oplus \partial(2B) \oplus \partial(4B) \tag{5}$$

Theorem

$$B \oplus B = B \oplus \partial(B) \tag{6}$$

Algorithms

- There are algorithms that have a computation time that decreases with the size!
- Openings are not always more “expensive” than erosions!

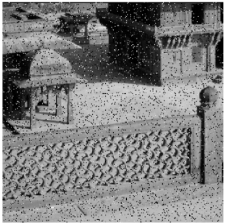


Filters

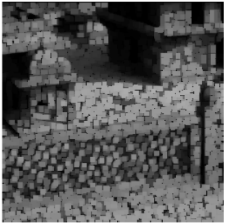
There are many filters:

- median,
- composition of openings,
- composition of openings and closings,
- area openings,
- openings by attribute,
- etc.

Median



Original image + noise

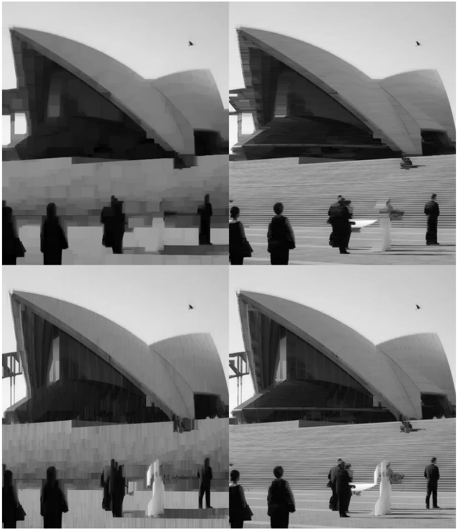


Opening with a 5×5 square

Butterworth low-pass filter

5×5 median

Supremum of openings



$\gamma_{mH \oplus nV}(f)$, $\gamma_{mH}(f)$, $\gamma_{nV}(f)$ and $\gamma_{mH}(f) \bigvee \gamma_{nV}(f)$

Algebraic filters

Filtres morphologiques

Definition

An *algebraic filter* is defined as an increasing and idempotent operator:

$$\psi \text{ is an algebraic } \mathit{filter} \Leftrightarrow \forall f, g \begin{cases} f \leq g \Rightarrow \psi(f) \leq \psi(g) \\ \psi(\psi(f)) = \psi(f) \end{cases} \tag{7}$$

Definition

An *algebraic opening* is an algebraic filter + anti-extensivity property:

$$\forall f, g,\ f \leq g \Rightarrow \psi(f) \leq \psi(g) \tag{8}$$

$$\forall f,\ \psi(\psi(f)) = \psi(f) \tag{9}$$

$$\forall f,\ \psi(f) \leq f \tag{10}$$

Structural theorm

Let ψ_1 and ψ_2 be two filters such that $\psi_1 \geq I \geq \psi_2$ (for example, ψ_1 is a closing and ψ_2 an opening).

Theorem

[Structural theorem] *Let ψ_1 and ψ_2 be two filters with $\psi_1 \geq I \geq \psi_2$*

$$\psi_1 \geq \psi_1\psi_2\psi_1 \geq (\psi_2\psi_1 \vee \psi_1\psi_2) \geq (\psi_2\psi_1 \wedge \psi_1\psi_2) \geq \psi_2\psi_1\psi_2 \geq \psi_2 \tag{11}$$

$$\psi_1\psi_2,\ \psi_2\psi_1,\ \psi_1\psi_2\psi_1,\ \psi_2\psi_1\psi_2 \textit{ are filters} \tag{12}$$

Note that $\psi_1\psi_2$ and $\psi_2\psi_1$ are not ordered.

Other filters: alternate sequential filters

Alternate sequential filters

If γ_i (ϕ_i) is an opening (resp. a closing) of size i and I is the identity operator (i.e. $I(f) = f$). Then these filters are all ordered:

$$\forall i,j \in \mathbb{N},\ \ i \leq j,\ \ \ \gamma_j \leq \gamma_i \leq I \leq \phi_i \leq \phi_j, \tag{13}$$

Then we define a series of operators according to:

$$\begin{aligned} m_i &= \gamma_i \phi_i, & r_i &= \phi_i \gamma_i \phi_i, \\ n_i &= \phi_i \gamma_i, & s_i &= \gamma_i \phi_i \gamma_i. \end{aligned}$$

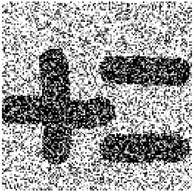
Definition

[Alternate sequential filters] For any $i \in \mathbb{N}$, the following filters are named alternate sequential filters of size i

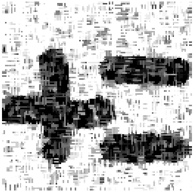
$$M_i = m_i m_{i-1} \ldots m_2 m_1 \qquad R_i = r_i r_{i-1} \ldots r_2 r_1 \tag{14}$$

$$N_i = n_i n_{i-1} \ldots n_2 n_1 \qquad S_i = s_i s_{i-1} \ldots s_2 s_1 \tag{15}$$

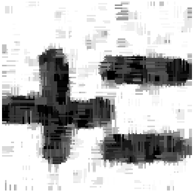
Alternate sequential filters



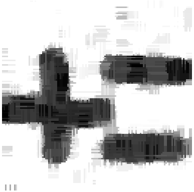
(a) Image f



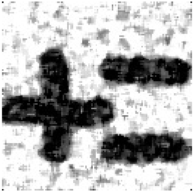
(b) $M_1(f)$



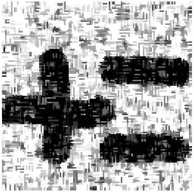
(c) $M_2(f)$



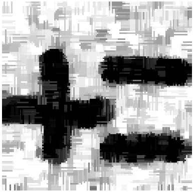
(d) $M_3(f)$



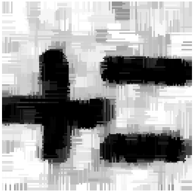
(e) Median 5×5



(f) $N_1(f)$

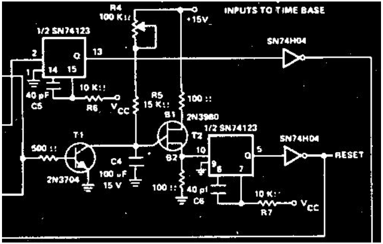


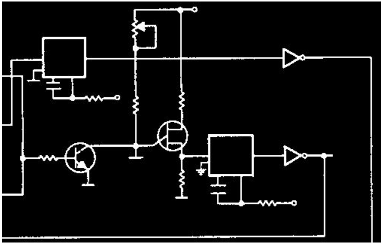
(g) $N_2(f)$



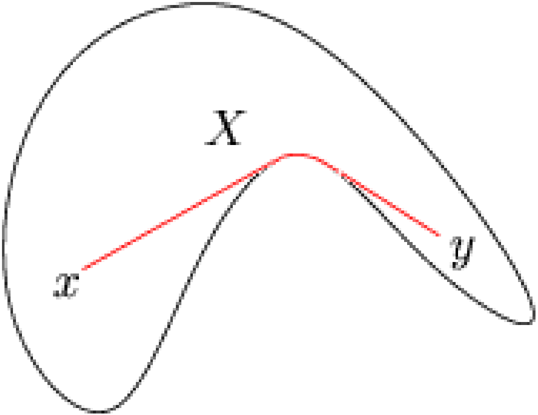
(h) $N_3(f)$

Area opening

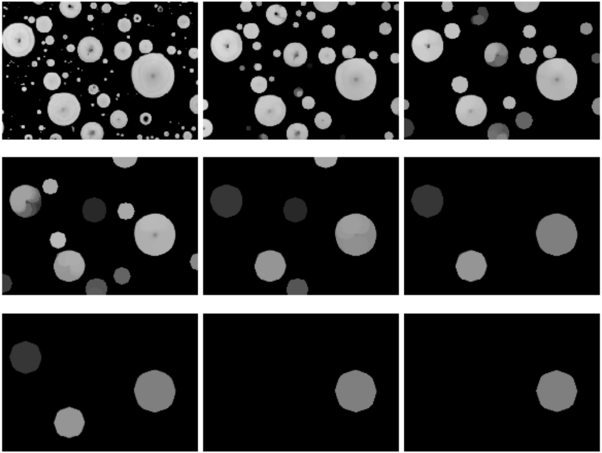




Geodesy



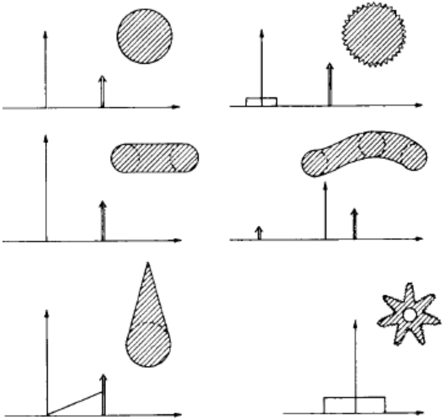
Grascale granulometries



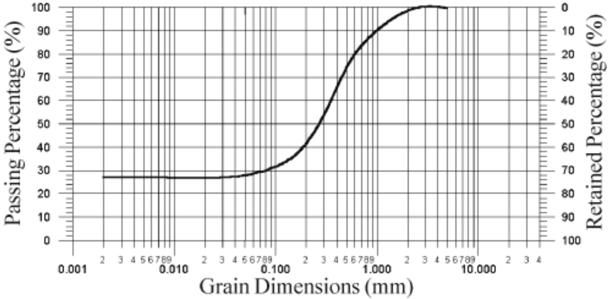
Granulometries

Definition

[Pattern Spectrum] $PS(X, B, r) = -\frac{d}{dr}\sharp(X \circ rB)$



Granulometric curve as a tool for classification

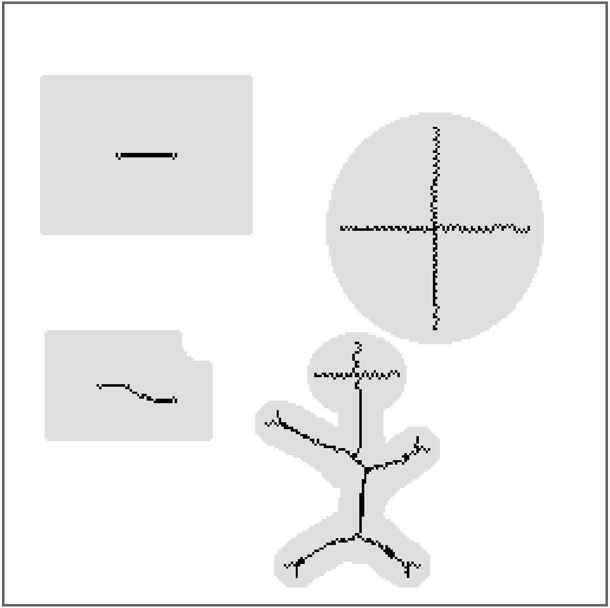


Clay	Silt	Fine sand	Coarse sand	Stoned soil
28.0%	8.1%	43.8%	19.9%	0.2%

<0.005mm 0.05mm 0.42mm >2.00mm

Figure 1. Granulometric curve of the soil mixture

Skeleton - medial axis



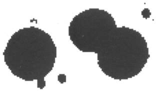
Distance map

Several definitions of distance (Euclidean distance is most common but not the easiest to compute).

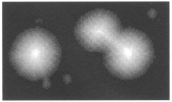
Definition

[Distance map] $\forall x \in X$

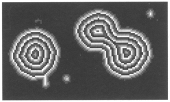
$$D_X(x) = \min d(x, X^c)$$



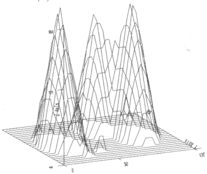
(a) Binary image of cells.



(b) Rounded Euclidean distance function on *(a)*.

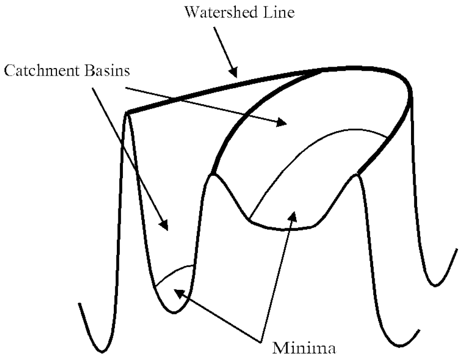


(c) Distance function modulo 4.

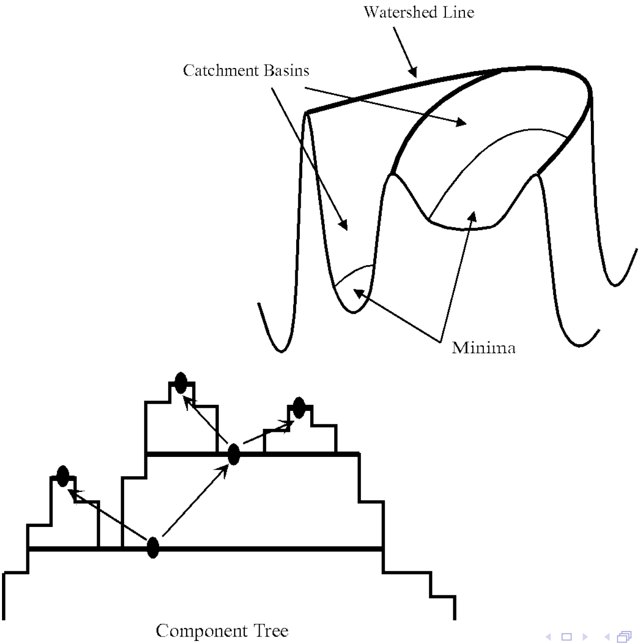


(d) Topographic representation of *(b)*.

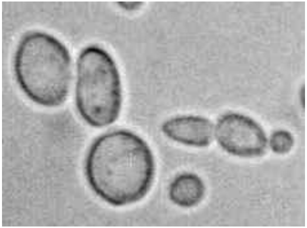
Watershed



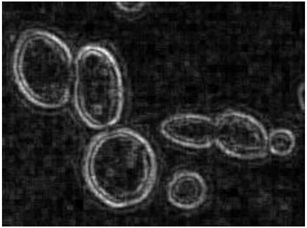
Watershed



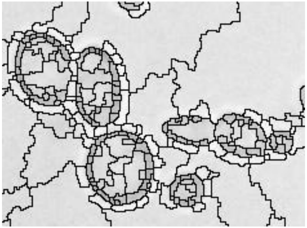
Watershed



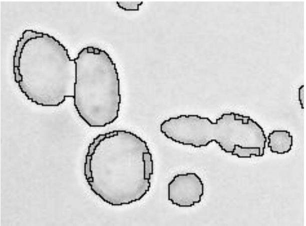
Original image



Gradient



Watersehd



Multiscale watershed

What about color images?

Paper: **A new approach to morphological color image processing** by G. Louverdis, M.I. Vardavoulia, I. Andreadis, Ph. Tsalides, 2002.

Abstract

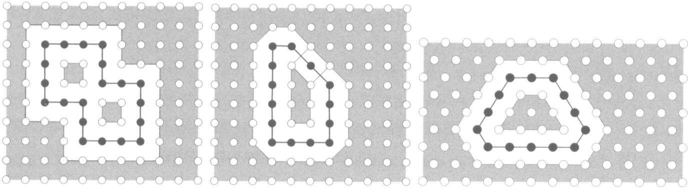
This paper presents a new approach to the generalization of the concepts of grayscale morphology to color images. A new vector ordering scheme is proposed, infimum and supremum operators are defined, and the fundamental vector morphological operations are extracted....

Topology

Theorem

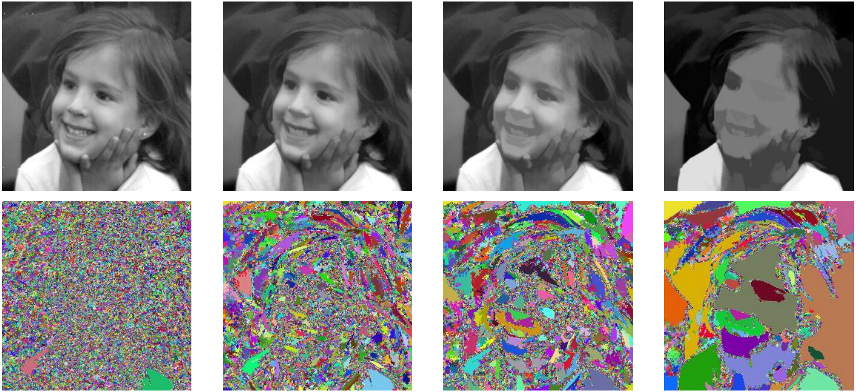
Jordan. Any simple closed curve (a closed curve that does not self-intersect) divides the plane into two distinct regions which are connected within themselves: one is of finite extent and the other not.

In the discrete case, this property is not true by default.



Connectivity

Levelings $\Rightarrow$ Flaten the image, which changes the connectivity



Trends

- Application to other types of data
- Segmentation
- Better algorithms
- Connectivity
- Use of graphs or trees
- Many applications

For further reading

Najman and Talbot (editors).
Mathematical Morphology.
Wiley, 2010.

P. Soille.
Morphological Image Analysis: Principles and Applications.
Springer, 1999.

H. Heijmans,
Morphological Image Operators.
Academic Press, 1994.